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2005 J. Phys. A: Math. Gen. 38 4999

(http://iopscience.iop.org/0305-4470/38/22/021)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 4999–5009 doi:10.1088/0305-4470/38/22/021

Construction of singular perturbations by the method of rigged Hilbert spaces

Volodymyr Koshmanenko

Institute of Mathematics, Tereshchenkivs'ka 3, Kyiv 01601, Ukraine

E-mail: kosh@imath.kiev.ua

Received 15 November 2004, in final form 7 April 2005
Published 18 May 2005
Online at stacks.iop.org/JPhysA/38/4999

Abstract
A new approach in the singular perturbations theory is proposed. Let A be a self-adjoint unbounded operator corresponding to the free Hamiltonian of some physical system in the state space $\mathcal{H}_0$ and $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ be the rigged Hilbert space associated with A, $\operatorname{dom} A = \mathcal{H}_+$ in the graph norm. Then a singular perturbation of A is defined as the unique self-adjoint operator $\breve{A}$ in $\mathcal{H}_0$ associated with a new rigged Hilbert space $\breve{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \breve{\mathcal{H}}_+$, where $\breve{\mathcal{H}}_+ = \operatorname{dom} \breve{A}$ is closed in $\mathcal{H}_+$ and densely embedded in $\mathcal{H}_0$ (such a kind of subspaces usually appears as a null space for a singular perturbant). We find the connections between A and $\breve{A}$ and investigate the properties of $\breve{A}$. In particular, we show that operators A and $\breve{A}$ are different on an infinite-dimensional subspace even in the case of a rank-1 singular perturbation ($\operatorname{codim} \breve{\mathcal{H}}_+ = 1$).

PACS numbers: 02.30.Tb, 03.65.Db
Mathematics Subject Classification: 47A10, 47A55

1. Introduction

Let $\mathcal{H}_0$ be a Hilbert space with the inner product $(\cdot,\cdot)_0$ and the norm $\|\cdot\|_0$ (the state space of the free physical system). And let $A = A^*$ be a positive unbounded self-adjoint operator in $\mathcal{H}_0$ corresponding to the free Hamiltonian. A trivial fact is that $\operatorname{dom} A \equiv \mathcal{D}(A)$ constitutes a Hilbert space $\mathcal{H}_+$ in the graph norm. This space is densely embedded in $\mathcal{H}_0$; write $\mathcal{H}_0 \sqsupset \mathcal{H}_+$. Let $\mathcal{H}_-$ be defined as the conjugate space to $\mathcal{H}_+$ with respect to $\mathcal{H}_0$. Then the triplet $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ is called the rigged (or equipped) Hilbert space. This triplet is uniquely associated with the operator A (for details see [6, 7, 20]).

Let a linear domain $\mathcal{D} \subset \mathcal{H}_+$ be dense in $\mathcal{H}_0$. One can think that $\mathcal{D} = \operatorname{Ker} T$, i.e., $\mathcal{D}$ is a null space of a singular perturbant $T : \mathcal{H}_+ \to \mathcal{H}_-$. According to the recognized conception in the singular perturbations theory [1–5, 8–23], the perturbed operator corresponding to a formal sum $A \tilde{+} T$ is defined as one of the self-adjoint extensions of the symmetric operator $\dot{A} := A|\mathcal{D}$.

In this paper, we propose a new approach for the construction of the singularly perturbed operator. Starting with an orthogonal decomposed $\mathcal{D}(A) = \mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$, such that $\mathcal{M}_+$ is dense in $\mathcal{H}_0$ (one should identify $\mathcal{M}_+$ with the above domain $\mathcal{D}$), we introduce the new rigged Hilbert chain $\breve{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \breve{\mathcal{H}}_+ \equiv \mathcal{M}_+$. Then we define the perturbed operator, which is denoted by $\breve{A}$, as one uniquely associated with the latter triplet. Thus, we extend the usual class of singularly perturbed operators. Besides all self-adjoint extensions of the symmetric operator $\dot{A} := A|\mathcal{D} \equiv A \mid \mathcal{M}_+$ we add to this class the operator $\breve{A}$ with the domain $\mathcal{D}(\breve{A}) = \mathcal{M}_+$. From our point of view, the choice $\breve{A}$ more adequately reflects the physical idea about a hard core of a singular interaction.

The arising mathematical problems are to study the properties of $\breve{A}$ and establish the connections between A and $\breve{A}$.

2. Some background on rigged Hilbert spaces

Here we recall some facts about rigging spaces and A-scales of Hilbert spaces (for more details see [6, 7]).

By definition a triplet of Hilbert spaces

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+, \tag{2.1}$$

is called the rigged (or equipped) Hilbert space if the following conditions are fulfilled: (a) both above embedding are continuous and dense (a symbol $\sqsupset$ just denotes the dense embedding), (b) the norms in $\mathcal{H}_-$, $\mathcal{H}_0$ and $\mathcal{H}_-$ are subjected to the inequalities

$$\|\cdot\|_- \leqslant \|\cdot\|_0 \leqslant \|\cdot\|_+,$$

(c) spaces $\mathcal{H}_-$ and $\mathcal{H}_+$ are mutually conjugated with respect to $\mathcal{H}_0$.

The latter condition means that each vector $\varphi \in \mathcal{H}_+$ generates in the inner product in $\mathcal{H}_0$ a linear functional $l_\varphi(f) := (f, \varphi)_0$, $f \in \mathcal{H}_0$, which has a continuous extension onto the whole space $\mathcal{H}_-$. Thus, the positive norm $\|\varphi\|_+$ may be calculated as follows, $\|\varphi\|_+ = \sup_{\|f\|_-=1} |(f, \varphi)_0|$. By the Riesz theorem $l_\varphi(f) = (f, \varphi^*)_-$ with a some $\varphi^* \in \mathcal{H}_-$. Therefore $\|\varphi\|_+ = \|\varphi^*\|_-$ and the mapping

$$D_{-,+} : \mathcal{H}_+ \ni \varphi \to \varphi^* \in \mathcal{H}_-$$

is isometric (in fact unitary). On the other hand $\mathcal{H}_-$ coincides with the completion of $\mathcal{H}_0$ with respect to the negative norm $\|f\|_- := \sup_{\|\varphi\|_+=1} |(\varphi, f)_0|$, $\varphi \in \mathcal{H}_+$. In turn, given $\mathcal{H}_0$ and $\mathcal{H}_-$ an element φ from $\mathcal{H}_0$ belongs to $\mathcal{H}_+$ iff the linear functional $l_\varphi(f) = (f, \varphi)_0$, $f \in \mathcal{H}_0$ is continuous on $\mathcal{H}_-$. Thus there exists the dual inner product between $\mathcal{H}_+$ and $\mathcal{H}_-$, which we denote by $\langle \omega, \varphi \rangle_{-,+} = \overline{\langle \varphi, \omega \rangle}_{+,-}$, $\omega \in \mathcal{H}_-$, $\varphi \in \mathcal{H}_+$.

The operators

$$D_{-,+} : \mathcal{H}_+ \to \mathcal{H}_- \qquad \text{and} \qquad I_{+,-} = D_{-,+}^{-1} : \mathcal{H}_- \to \mathcal{H}_+$$

are called the canonical unitary isomorphisms between $\mathcal{H}_-$ and $\mathcal{H}_+$. They satisfy the relations:

$$(f, \varphi)_0 = (f, D_{-,+}\varphi)_-, \qquad \langle \omega, \varphi \rangle_{-,+} = (I_{+,-}\omega, \varphi)_+,$$
$$f \in \mathcal{H}_0, \qquad \omega \in \mathcal{H}_-, \qquad \varphi \in \mathcal{H}_+,$$

and

$$\|\varphi\|_+ = \|D_{-,+}\varphi\|_- = \|\varphi^*\|_-, \qquad \|\omega\|_- = \|I_{+,-}\omega\|_+,$$
$$\varphi \in \mathcal{H}_+, \qquad \omega = \varphi^* \in \mathcal{H}_-,$$

where φ^* was defined above.

There exists the well-known connection between rigged Hilbert spaces of the type (2.1) and self-adjoint operators A in $\mathcal{H}_0$. This connection is fixed by $D_{-,+}$ and the condition $\mathcal{D}(A) = \mathcal{H}_+$. Indeed, let us consider the operator

$$L_A := D_{-,+}|\mathcal{H}_{++}, \qquad \text{where} \qquad \mathcal{H}_{++} \equiv \mathcal{D}(L_A) := \{\varphi \in \mathcal{H}_+ | D_{-,+}\varphi \in \mathcal{H}_0\}.$$

Obviously L_A is symmetric in $\mathcal{H}_0$ since for all $\varphi, \psi \in \mathcal{D}(L_A)$,

$$\begin{aligned}(L_A\varphi, \psi)_0 &= (D_{-,+}\varphi, \psi)_0 = \langle \varphi^*, \psi \rangle_{-,+} = (\varphi, \psi)_+ \\ &= \langle \varphi, \psi^* \rangle_{+,-} = (\varphi, D_{-,+}\psi)_0 = (\varphi, L_A\psi)_0.\end{aligned}$$

Therefore, L_A is self-adjoint in $\mathcal{H}_0$ because by the construction its range runs through the whole $\mathcal{H}_0$. By the construction L_A is positive. We define $A := L_A^{1/2}$. Clearly $\mathcal{D}(A) = \mathcal{H}_+$ due to $(L_A\varphi, \psi)_0 = \left(L_A^{1/2}\varphi, L_A^{1/2}\psi\right)_0 = (\varphi, \psi)_+$. Evidently also that $A \geqslant 1$ due to $\|\cdot\|_+ \geqslant \|\cdot\|$.

Vice versa, let $A = A^* \geqslant 1$ be a self-adjoint unbounded operator with domain $\mathcal{D}(A)$ in a Hilbert space $\mathcal{H}_0$. Using A one can easily construct the rigged Hilbert space $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$. We recall these constructions.

Define the space $\mathcal{H}_+$ as the domain $\mathcal{D}(A)$ endowed by the inner product $(\varphi, \psi)_+ := (A\varphi, A\psi)_0, \varphi, \psi \in \mathcal{D}(A)$. Then starting with the pre-rigged pair $\mathcal{H}_0 \sqsupset \mathcal{H}_+$ we extend it to the rigged Hilbert space (2.1) by the standard way. Thus, the following theorem is true.

Theorem 2.1. *Each rigged Hilbert space (2.1) is uniquely associated with the self-adjoint in $\mathcal{H}_0$ operator $A = A^* \geqslant 1$ such that $\mathcal{D}(A) = \mathcal{H}_+$ in the norm $\|\varphi\|_+ = \|A\varphi\|_0, \varphi \in \mathcal{D}(A)$.*

We remark that the above connection between rigged Hilbert spaces and self-adjoint operators admits the extension to the case of an arbitrary self-adjoint but not necessary positive operators A. To reach this, one has to put $\|\varphi\|_+ = \|(A + \mathrm{i})\varphi\|_0, \varphi \in \mathcal{D}(A)$.

Now we recall the construction of the infinite chain of Hilbert spaces $\{\mathcal{H}_\alpha \equiv \mathcal{H}_\alpha(A)\}_{\alpha\in\mathbb{R}}$, which extends the above rigged Hilbert space and is called the A-scale.

For each $\alpha > 0$ we define the Hilbert space $\mathcal{H}_\alpha \equiv \mathcal{H}_\alpha(A)$ which coincides, as a set, with the domain $\mathcal{D}(A^{\alpha/2})$ and which is the complete space with respect to the norm $\|\cdot\|_\alpha$ corresponding to the inner product

$$(\varphi, \psi)_\alpha := (A^{\alpha/2}\varphi, A^{\alpha/2}\psi)_0 \qquad \varphi, \psi \in \mathcal{D}(A^{\alpha/2}).$$

Then the space $\mathcal{H}_{-\alpha}$ appears as the completion of $\mathcal{H}_0$ in the negative norm

$$\|f\|_{-\alpha} := \|A^{-\alpha/2}f\|_0, \qquad f \in \mathcal{H}_0.$$

It is easy to see that each triplet

$$\mathcal{H}_{-\alpha} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_\alpha, \qquad \alpha > 0 \tag{2.2}$$

organizes the rigged Hilbert space associated with $A^{\alpha/2}$. In particular, according to the previous notations, $\mathcal{H}_+ = \mathcal{H}_2(A)$ and $\mathcal{H}_- = \mathcal{H}_{-2}(A)$.

Let $D_{-\alpha,\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_{-\alpha}$ denote the operator of canonical unitary isomorphism for the rigged triplet (2.2). This follows that $D_{-\alpha,\alpha} = (A^{\alpha/2})^{\mathrm{cl}}(A^{\alpha/2}) \equiv D_{-\alpha,0}D_{0,\alpha}$, where cl stands for a closure of the mapping $A^{\alpha/2} \equiv D_{0,\alpha} : \mathcal{H}_0 \to \mathcal{H}_{-\alpha/2}$. For $\alpha = 2$, we have: $D_{0,2} \equiv A : \mathcal{H}_2 \to \mathcal{H}_0$ and $D_{-2,0} \equiv A^{\mathrm{cl}} : \mathcal{H}_0 \to \mathcal{H}_{-2}$.

3. The rigged Hilbert spaces approach

Let a singular perturbation of the starting Hamiltonian in $\mathcal{H}_0$ be given by an unclosable quadratic form γ with a domain $Q(\gamma) \subset \operatorname{dom} A$. Assume that γ belongs to the so-called $\mathcal{H}_{-2}$-class [3, 20, 22]. This means, in particular, that γ is zero almost everywhere in $\mathcal{H}_0$ but it is closable in $\mathcal{H}_+ = \mathcal{D}(A)$. Therefore, there exists the operator $T \equiv T_\gamma : \mathcal{H}_+ \to \mathcal{H}_-$, which is associated with γ, such that the subspace $\mathcal{M}_+ := \mathrm{Ker}\, T_\gamma$ is dense in $\mathcal{H}_0$ and the subspace $\mathcal{N}_- := (\mathrm{Ran}\, T)^{\mathrm{cl},-}$ has a zero intersection with $\mathcal{H}_0$, where cl,- stands for the closure in $\mathcal{H}_-$.

For the construction of the perturbed operator corresponding to the formal sum $A \tilde{+} T$, usually one introduces the symmetric restriction

$$\dot{A} := A \mid \mathcal{M}_+$$

and considers the family of all self-adjoint extensions A_α of $\dot{A}$ (α stands for a parameter of extensions) in the role of singular perturbations of A caused by γ. For example, the one-dimensional Schrödinger operator $-\Delta_{\lambda,\alpha}$ with the δ_y-potential is fixed by the boundary condition of the type: $\varphi'(y+0) - \varphi'(y-0) = \alpha\varphi(y)$, $y \in \mathbb{R}$ (see e.g. [1]).

Here we propose another method. We interpret a singular perturbation as an interaction with an absolutely hard core (or an impenetrable screen) situated in a small physical volume of zero Lebesgue measure. So, inside of this volume, which corresponds to the subspace $\mathcal{N}_+ := \mathcal{H}_+ \ominus \mathcal{M}_+$, the action of the perturbed Hamiltonian is absolutely unknown, and we propose to cut off this subspace from Hamiltonian's domain. To the point, on the strength of singularity, the values of T on vectors $\varphi \in \mathrm{dom}\, T \cap \mathcal{N}_+$ have infinite norms in the sense of the space $\mathcal{H}_0$. Of course, it is natural to preserve the Hamiltonian without any changes on the domain $\mathrm{Ker}\, T$. However, in general it is impossible if we assume that in the physical space an impenetrable object (a hard core) appears, even if it is situated in a zero measure volume. In fact, the corresponding perturbed dynamics should occur outside of such volume and therefore the operator's domain of the Hamiltonian is subjected to a certain contraction, which is fulfilled by the projection onto $\mathcal{M}_+$ (see formula (4.11)). Thus, we may take the subspace $\mathrm{Ker}\, T$ as the domain of a new self-adjoint operator $\breve{A}$ and interpret it as a perturbation of A. We will show that such operator is uniquely defined by the condition $\mathcal{D}(\breve{A}) = \mathrm{Ker}\, T$. We propose to consider the operator $\breve{A}$ as an additional version in the definition of singular perturbations for A.

Apparently, in the same way starting with operators A_α one can introduce a family of operators $\breve{A}_\alpha$ with $\mathcal{D}(\breve{A}_\alpha) \subseteq \mathrm{Ker}\, T$, which may be considered in a role of singular perturbations for A too.

Here we remark the following important feature of our constructions. Always the operators $\breve{A}$, A differ on the infinite-dimensional subspace (see proposition 5.1); however their square powers $\breve{A}^2$, A^2 coincide on the dense subset in $\mathcal{H}_0$.

We formulate our main result as follows:

Theorem 3.1. *Let A be a positive self-adjoint operator with a domain $\mathcal{D}(A)$ in a Hilbert space $\mathcal{H}_0$. Denote $\mathcal{D}(A) = \mathcal{H}_+$ in the graph norm. Let a singular perturbant of A be given by some operator $T : \mathcal{H}_+ \to \mathcal{H}_-$ such that the subspace $\mathcal{M}_+ := \mathrm{Ker}\, T$ is dense in $\mathcal{H}_0$ (for a dense continuous embedding we use notation $\mathcal{M}_+ \sqsubset \mathcal{H}_0$). Consider the new rigged Hilbert space*

$$\breve{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \breve{\mathcal{H}}_+, \qquad \text{where} \quad \breve{\mathcal{H}}_+ \equiv \mathcal{M}_+, \tag{3.1}$$

and $\breve{\mathcal{H}}_-$ is the conjugate space to $\mathcal{M}_+$. Assume that a codimension of $\mathcal{M}_+$ in $\mathcal{H}_+$ is finite, i.e.,

$$\dim \mathcal{N}_+ < \infty, \qquad \text{where} \quad \mathcal{N}_+ := \mathcal{H}_+ \ominus \mathcal{M}_+. \tag{3.2}$$

Then the self-adjoint operator $\breve{A}$ such that $\mathcal{D}(\breve{A}) = \mathcal{M}_+$ is uniquely associated with the triplet (3.1). Moreover, this operator admits the following explicit description in terms of the mapping

$$L : P_{\mathcal{M}_+}\varphi \to A^2\varphi, \qquad \varphi \in \mathcal{D}(A^2),$$

where $P_{\mathcal{M}_+}$ stands for the orthogonal projection onto $\mathcal{M}_+$ in $\mathcal{H}_+$. Namely

$$\breve{A} = L^{1/2}.$$

Remark. Theorem 3.1 is true also if $\dim \mathcal{N}_+ = \infty$ but the following condition holds:

$$(\mathcal{N}_-)^{\mathrm{cl},--} \cap \mathcal{H}_- = \mathcal{N}_-, \tag{3.3}$$

where $\mathcal{N}_- := D_{-,+}\mathcal{N}_+$, and cl, $--$ stands for a closure in $\mathcal{H}_{--} \equiv \mathcal{H}_{-4}(A)$ (see the previous section for notations).

To the end of this section we discuss the question, when a subspace $\mathcal{M}_+$ from $\mathcal{H}_+$ is dense in $\mathcal{H}_0$?

Let a rigged Hilbert space $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ be given. Assume that the positive space $\mathcal{H}_+$ is decomposed into an orthogonal sum $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$. There exists a simple criterion ensuring the dense embedding $\mathcal{H}_0 \sqsupset \mathcal{M}_+$.

Theorem 3.2 [2]. *Let $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$. A closed in $\mathcal{H}_+$ subspace $\mathcal{M}_+$ is dense in $\mathcal{H}_0$ iff the subspace $\mathcal{N}_- := D_{-,+}\mathcal{N}_+$ has a zero intersection with $\mathcal{H}_0$,*

$$\mathcal{H}_0 \sqsupset \mathcal{M}_+ \quad \Leftrightarrow \quad \mathcal{N}_- \cap \mathcal{H}_0 = \{0\}. \tag{3.4}$$

Equivalently,

$$\mathcal{H}_0 \sqsupset \mathcal{M}_+ \quad \Leftrightarrow \quad \mathcal{N}_0 \cap \mathcal{H}_+ = \{0\}, \qquad \text{where} \quad \mathcal{N}_0 := D_{0,+}\mathcal{N}_+. \tag{3.5}$$

Proof. Let us prove (3.4). Let $\mathcal{N}_- \cap \mathcal{H}_0 = \{0\}$ and assume that there exists a vector $0 \neq \psi \in \mathcal{H}_0$, $\psi \perp \mathcal{M}_+$. Since $\mathcal{M}_+$ is a subspace of $\mathcal{H}_+$ and due to $\psi \in \mathcal{H}_-$ we have

$$0 = (\psi, \mathcal{M}_+)_0 = \langle \psi, \mathcal{M}_+ \rangle_{-,+} = (I_{+,-}\psi, \mathcal{M}_+)_+.$$

Therefore $I_{+,-}\psi \in \mathcal{N}_+$. This means that $\psi \in \mathcal{N}_-$, that is a contradiction to the starting assumption. Vice versa, if the subspace $\mathcal{M}_+$ is dense in $\mathcal{H}_0$ then assumption that there exists a vector $0 \neq \omega \in \mathcal{N}_- \cap \mathcal{H}_0$ leads to contradiction too. Indeed, since $\mathcal{N}_- = D_{-,+}\mathcal{N}_+$ we have,

$$\langle \omega, \mathcal{M}_+ \rangle_{-,+} = (\omega, \mathcal{M}_+)_0 = (I_{+,-}\omega, \mathcal{M}_+)_+ = 0$$

that is a contradiction with $\mathcal{M}_+ \sqsubset \mathcal{H}_0$ since $0 \neq \omega \in \mathcal{H}_0$. □

Corollary 3.3. *Under condition (3.3) the set $\tilde{\mathcal{M}}_+ := Ker\ T \cap \mathcal{D}(A^2)$ is dense in $\mathcal{H}_0$, and*

$$\breve{A}^2 | \tilde{\mathcal{M}}_+ = A^2 | \tilde{\mathcal{M}}_+.$$

4. The construction of the operator $\breve{A}$

Let $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ be the rigged Hilbert space associated with a self-adjoint in $\mathcal{H}_0$ operator $A \geqslant 1$. So, $\mathcal{H}_+ = \mathcal{H}_2(A) = \mathcal{D}(A)$ in the norm $\|\cdot\|_+ = \| A \cdot \|_0$, and A^2 coincides with the restriction of $D_{-,+} : \mathcal{H}_+ \to \mathcal{H}_-$ onto $\mathcal{H}_{++} \equiv \mathcal{H}_4(A)$,

$$A^2 = D_{-,+} | \mathcal{H}_{++}.$$

Assume the positive space $\mathcal{H}_+$ is decomposed into an orthogonal sum $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+$ in such a way that the subspace $\mathcal{M}_+$ is dense in $\mathcal{H}_0$, $\mathcal{H}_0 \sqsupset \mathcal{M}_+$. Consider a new rigged Hilbert space

$$\breve{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \breve{\mathcal{H}}_+, \qquad \text{where} \quad \breve{\mathcal{H}}_+ \equiv \mathcal{M}_+. \tag{4.1}$$

Here we will describe the construction of the self-adjoint operator $\breve{A}$ associated with the chain (4.1) in such a way that the domain $\mathcal{D}(\breve{A}) = \breve{\mathcal{H}}_+$.

We recall that the negative space $\breve{\mathcal{H}}_-$ is defined as a completion of $\mathcal{H}_0$ with respect to the new negative norm:

$$\| f \|\breve{}_- := \sup_{\|\varphi\|_+ = 1} |(f, \varphi)_0|, \qquad \varphi \in \mathcal{M}_+. \tag{4.2}$$

It is clear that for any fixed $f \in \mathcal{H}_0$

$$\|f\|\breve{}_- \leqslant \|f\|_-, \tag{4.3}$$

where we recall that

$$\|f\|_- := \sup_{\|\varphi\|_+=1} |(f,\varphi)_0|, \qquad \varphi \in \mathcal{H}_+.$$

Thus, the space $\mathcal{H}_0$ is densely embedded both in $\mathcal{H}_-$ and $\breve{\mathcal{H}}_-$. Looking at (4.3) one can naively think that $\mathcal{H}_-$ is embedded in $\breve{\mathcal{H}}_-$ as a subset; however this is not true.

Proposition 4.1. *A closure of the identical mapping (this mapping is bounded due to (4.3))*

$$J : \mathcal{H}_- \ni f \to f \in \breve{\mathcal{H}}_-, \qquad f \in \mathcal{H}_0,$$

has a non-trivial zero-subset:

$$\operatorname{Ker} J^{\mathrm{cl}} = \mathcal{N}_-, \qquad \text{where} \quad \mathcal{N}_- = I_{-,+}\mathcal{N}_+, \quad (\mathrm{cl} = \mathrm{closure}).$$

Proof. Let a sequence f_n, $f_n \in \mathcal{H}_0$, converges in $\mathcal{H}_-$ to some element $\eta_- \in \mathcal{N}_-$. Then by (4.3) this sequence is convergent in $\breve{\mathcal{H}}_-$ too. Moreover

$$(f_n,\varphi)_0 = \langle f_n,\varphi\rangle_{-,+} \to \langle \eta_-,\varphi\rangle_{-,+} = 0, \qquad \varphi \in \mathcal{M}_+ \equiv \breve{\mathcal{H}}_+,$$

since $\mathcal{N}_- \perp \mathcal{M}_+$. This means that $f_n \to 0$ in $\breve{\mathcal{H}}_-$. Therefore $\eta_- \in \operatorname{Ker} J^{\mathrm{cl}}$. □

In fact, the restriction of J^{cl} onto subspace $\mathcal{M}_- := D_{-,+}\mathcal{M}_+$ is a unitary operator. It follows from

Proposition 4.2. *For each $f \in \mathcal{H}_0$,*

$$\|f\|\breve{}_- = \|P_{\mathcal{M}_-} f\|_-,$$

where $P_{\mathcal{M}_-}$ stands for the orthogonal projection onto $\mathcal{M}_-$ in $\mathcal{H}_-$.

Proof. If $\varphi \in \mathcal{M}_+$, then we have

$$(f,\varphi)_0 = \langle f,\varphi\rangle_{-,+} = \langle P_{\mathcal{M}_-} f,\varphi\rangle_{-,+},$$

where we used $\mathcal{M}_- \perp \mathcal{N}_+$ and (4.2). □

Thus, the space $\mathcal{H}_-$ does not belong to $\breve{\mathcal{H}}_-$ as a part,

$$\breve{\mathcal{H}}_- \not\supseteq \mathcal{H}_-, \tag{4.4}$$

in spite of that $\breve{\mathcal{H}}_+ \equiv \mathcal{M}_+$ is a proper part of $\mathcal{H}_+$ and inequality (4.3) holds.

By proposition 4.2 the spaces $\breve{\mathcal{H}}_-$ and $\mathcal{M}_-$ are unitary equivalent. We also remark that

$$P_{\mathcal{M}_-} f \neq 0 \quad \text{for any } f \in \mathcal{H}_0. \tag{4.5}$$

Indeed, $P_{\mathcal{M}_-} f = 0$ means that $f \in \mathcal{N}_-$; however $\mathcal{N}_- \cap \mathcal{H}_0 = \{0\}$ due to $\mathcal{H}_0 \sqsupset \mathcal{M}_+$ (see theorem 3.2).

Let $\breve{D}_{-,+} : \breve{\mathcal{H}}_+ \to \breve{\mathcal{H}}_-$ denote the canonical unitary isomorphism in the rigged Hilbert space (4.1).

Proposition 4.3. *The subspace $\tilde{\mathcal{M}}_+ := \mathcal{M}_+ \cap \mathcal{H}_{++}$ is closed in $\mathcal{H}_{++}$. Moreover, the subspace $\tilde{\mathcal{M}}_+$ is dense in $\mathcal{H}_0$,*

$$\mathcal{H}_0 \sqsupset \tilde{\mathcal{M}}_+, \tag{4.6}$$

if condition (3.3) is fulfilled. In particular, $\tilde{\mathcal{M}}_+$ is dense in $\mathcal{H}_0$ if the dimension of $\mathcal{N}_+$ is finite.

Proof. Let a sequence $\varphi_n \in \tilde{\mathcal{M}}_+$ be convergent in $\mathcal{H}_{++}$: $\varphi_n \to \varphi \in \mathcal{H}_{++}$. Then it is convergent in $\mathcal{H}_+$ too, due to $\|\cdot\|_+ \leqslant \|\cdot\|_{++}$. Therefore $\varphi \in \mathcal{M}_+$ since $\mathcal{M}_+$ is closed in $\mathcal{H}_+$. Thus, $\tilde{\mathcal{M}}_+$ is closed in $\mathcal{H}_{++}$ too.

Further, using the definition of $\tilde{\mathcal{M}}_+$ in the form

$$\tilde{\mathcal{M}}_+ = \{\varphi \in \mathcal{H}_{++} | (\varphi, \psi)_+ = 0, \psi \in \mathcal{N}_+\},$$

by properties of the A -scale we have,

$$(\varphi, \psi)_+ = \langle \varphi, \omega \rangle_{+,-} = \langle \varphi, \omega \rangle_{++,--}, \qquad \text{where} \quad \omega = D_{-,+}\psi, \quad \psi \in \mathcal{N}_+.$$

This implies that

$$\tilde{\mathcal{N}}_- = (\mathcal{N}_-)^{\mathrm{cl},--}, \tag{4.7}$$

where

$$\tilde{\mathcal{N}}_- := \{\omega \in \mathcal{H}_{--} | \langle \varphi, \omega \rangle_{++,--} = 0, \varphi \in \tilde{\mathcal{M}}_+\}.$$

Further, under condition (3.3), $\tilde{\mathcal{N}}_- \cap \mathcal{H}_0 = \{0\}$. Therefore $\mathcal{H}_0 \sqsupset \tilde{\mathcal{M}}_+$ due to theorem 3.2. Finally we note that condition (3.3) is automatically fulfilled, if $\dim \mathcal{N}_0 = \dim \mathcal{N}_+ < \infty$. □

Proposition 4.4. *The mappings $\check{D}_{-,+}$, $D_{-,+}$ coincide on the subspace $\tilde{\mathcal{M}}_+ = \mathcal{M}_+ \cap \mathcal{H}_{++}$ and moreover, they map $\tilde{\mathcal{M}}_+$ into $\mathcal{H}_0$:*

$$\check{D}_{-,+}\varphi = D_{-,+}\varphi \in \mathcal{H}_0, \varphi \in \tilde{\mathcal{M}}_+. \tag{4.8}$$

Proof. We recall that $\mathcal{H}_{++} \equiv \mathcal{H}_4(A) = \mathcal{D}(A^2)$. Therefore the vector $f := D_{-,+}\varphi = A^2\varphi \in \mathcal{H}_0$ for each $\varphi \in \mathcal{H}_{++}$.

Consider for a fixed $\varphi \in \tilde{\mathcal{M}}_+$ two functionals:

$$l_\varphi(\psi) := \langle D_{-,+}\varphi, \psi \rangle_{-,+}, \qquad \psi \in \mathcal{H}_+$$

and

$$\check{l}_\varphi(\psi) := \langle \check{D}_{-,+}\varphi, \psi \check{\rangle}_{-,+}, \qquad \psi \in \mathcal{M}_+.$$

Since $f = D_{-,+}\varphi \in \mathcal{H}_0$ the functional $l_\varphi(\psi)$ is continuous on $\mathcal{H}_0$, and $l_\varphi(\psi) = (f, \psi)_0 = (\varphi, \psi)_+$ for all $\psi \in \mathcal{M}_+$.

The functional $\check{l}_\varphi(\psi)$ is continuous on $\mathcal{H}_0$ too, since $\mathcal{M}_+ = \check{\mathcal{H}}_+$ and $\check{l}_\varphi(\psi) = (\varphi, \psi)_+ = \langle \varphi, \psi \rangle_{\mathcal{H}_{++},\mathcal{H}_0}$, $|\check{l}_\varphi(\psi)| \leqslant c\ \|\psi\|_0$, where $c = \|\varphi\|_{++}$. Therefore $\check{l}_\varphi(\psi) = (\check{f}, \psi)_0$ with some $\check{f} \in \mathcal{H}_0$.

We assert that $f = \check{f}$. Indeed, by the construction $(f, \psi)_0 = (\varphi, \psi)_+ = (\check{f}, \psi)_0$ for all $\psi \in \mathcal{M}_+$. This implies that the vectors f and $\check{f}$ coincide, since the subspace $\mathcal{M}_+$ is dense in $\mathcal{H}_0$. This completes the proof. □

Proposition 4.5. *Under condition (3.3) (or (3.2)) the subspace $\tilde{\mathcal{M}}_+$ is dense in $\mathcal{M}_+$.*

Proof. If $\phi \in \mathcal{M}_+$ and $\phi \perp \tilde{\mathcal{M}}_+$, then $D_{-,+}\phi \perp \mathcal{N}_-$ and $D_{-,+}\phi \in \tilde{\mathcal{N}}_-$. Therefore $\phi \equiv 0$ since $\tilde{\mathcal{N}}_- = \mathcal{N}_-$ due to (3.3). In more details, let $\mathcal{M}_+ = \tilde{\mathcal{M}}_+ \oplus \tilde{\mathcal{M}}_+^\perp$ and $\phi \in \tilde{\mathcal{M}}_+^\perp$. Then $\omega := D_{-,+}\phi \in \tilde{\mathcal{M}}_-^\perp$, where $\tilde{\mathcal{M}}_-^\perp = \tilde{\mathcal{M}}_- \ominus \tilde{\mathcal{M}}_-^\perp$. Therefore we have

$$\langle \omega, \tilde{\mathcal{M}}_+ \rangle_{-,+} = 0 = \langle \omega, \tilde{\mathcal{M}}_+ \rangle_{--,++} \quad \Rightarrow \quad \omega \in \tilde{\mathcal{N}}_- = \mathcal{N}_-.$$

But this is possible only if $\phi = 0$ since $\phi \in \mathcal{M}_+$ and $D_{-,+}\phi \perp \mathcal{N}_-$. □

Let us consider the operator

$$\dot{L} := D_{-,+}|\tilde{\mathcal{M}}_+ = \check{D}_{-,+}|\tilde{\mathcal{M}}_+ = A^2|\tilde{\mathcal{M}}_+. \tag{4.9}$$

Due to proposition 4.3 it is the closed densely defined symmetric operator in $\mathcal{H}_0$. We assert that the range of $\dot{L}$ is dense in $\breve{\mathcal{H}}_-$. Indeed, the range of $\dot{L}$ coincides with the subspace $\tilde{\mathcal{M}}_- = A^2\tilde{\mathcal{M}}_+ = A^2(\mathcal{M}_+ \cap \mathcal{H}_{++}) = \mathcal{M}_- \cap \mathcal{H}_0$, which is dense in $\breve{\mathcal{H}}_-$ due to $\breve{D}_{-,+} : \breve{\mathcal{H}}_+ \to \breve{\mathcal{H}}_-$ being the unitary operator.

Proposition 4.6. *The Friedrichs extension of the symmetric operator $\dot{L}$ can be defined as follows*

$$L_\infty \equiv L := \breve{D}_{-,+}|\mathcal{D}(L), \quad \text{with} \quad \mathcal{D}(L) := \{\varphi \in \mathcal{M}_+ | \breve{D}_{-,+}\varphi \in \mathcal{H}_0\}. \tag{4.10}$$

Proof. By proposition 4.5, the space $\breve{\mathcal{H}}_+$ coincides with the completion of $\tilde{\mathcal{M}}_+$ in the inner product $(\varphi, \psi)_{\breve{\mathcal{H}}_+} := (\dot{L}\varphi, \psi)_0 = (A\varphi, A\psi)_0 = (\varphi, \psi)_+, \varphi, \psi \in \tilde{\mathcal{M}}_+$. Therefore L is an extension of $\dot{L}$. Obviously the operator L is symmetric and its range is the whole space $\mathcal{H}_0$. This means that L is self-adjoint. By construction it is the Friedrichs extension of $\dot{L}$ since $\tilde{\mathcal{M}}_+ \sqsubset \mathcal{M}_+$. □

Now we give a more explicit description of the operator L.

Theorem 4.7. *The operator L defined by (4.10) admits the following representation:*

$$\mathcal{D}(L) = P_{\mathcal{M}_+}\mathcal{H}_{++}, \qquad LP_{\mathcal{M}_+}\varphi = A^2\varphi, \qquad \varphi \in \mathcal{D}(A^2) = \mathcal{H}_{++}, \tag{4.11}$$

where $P_{\mathcal{M}_+}$ stands for the orthogonal projection onto $\mathcal{M}_+$ in $\mathcal{H}_+$.

Proof. Let us show that the mapping

$$L : P_{\mathcal{M}_+}\varphi \to A^2\varphi, \qquad \varphi \in \mathcal{H}_{++}$$

is a symmetric operator in $\mathcal{H}_0$. Indeed, for each $\varphi, \psi \in \mathcal{H}_{++}$ we have

$$\begin{aligned}(LP_{\mathcal{M}_+}\varphi, P_{\mathcal{M}_+}\psi)_0 &= (A^2\varphi, P_{\mathcal{M}_+}\psi)_0 = \langle D_{-,+}\varphi, P_{\mathcal{M}_+}\psi\rangle_{-,+}\\ &= \langle P_{\mathcal{M}_-}D_{-,+}\varphi, P_{\mathcal{M}_+}\psi\rangle_{-,+} = \langle D_{-,+}P_{\mathcal{M}_+}\varphi, P_{\mathcal{M}_+}\psi\rangle_{-,+} = \langle P_{\mathcal{M}_+}\varphi, D_{-,+}P_{\mathcal{M}_+}\psi\rangle_{+,-}\\ &= \langle P_{\mathcal{M}_+}\varphi, D_{-,+}\psi\rangle_{+,-} = \langle P_{\mathcal{M}_+}\varphi, A^2\psi\rangle_{+,-} = (P_{\mathcal{M}_+}\varphi, LP_{\mathcal{M}_+}\psi)_0.\end{aligned}$$

This implies that L is self-adjoint since its range $\mathcal{R}(L) = \mathcal{R}(A^2) = \mathcal{H}_0$. Further, obviously $P_{\mathcal{M}_+}\tilde{\mathcal{M}}_+ = \tilde{\mathcal{M}}_+$. It follows that $\tilde{\mathcal{M}}_+ \subset \mathcal{D}(L)$ and $L\tilde{\mathcal{M}}_+ = \dot{L}\tilde{\mathcal{M}}_+ = A^2\tilde{\mathcal{M}}_+$ (see (4.10)). Therefore, the operator L defined by (4.11) coincides with the operator L in (4.10) since $\tilde{\mathcal{M}}_+$ is dense in $\mathcal{M}_+$ (see proposition 4.5). □

Finally we introduce the operator

$$\breve{A} := L^{1/2}.$$

We assert that the domain of $\breve{A}$ exactly coincides with the subspace $\mathcal{M}_+$, i.e.,

$$\mathcal{D}(\breve{A}) = \mathcal{M}_+. \tag{4.12}$$

Proposition 4.8. *(4.12) is true since the completion of the set $\mathcal{D}((\breve{A})^2) = \mathcal{D}(L)$ in the norm $\|\cdot\|_+ := \|L^{1/2}\cdot\|_0$ coincides with $\mathcal{M}_+$.*

Proof. Indeed, since $\tilde{\mathcal{M}}_+$ is dense in $\mathcal{M}_+$ we need to recall only that $(L\varphi, \psi)_0 = (\varphi, \psi)_+, \varphi, \psi \in \tilde{\mathcal{M}}_+$. Therefore, by the definition of L, we have $(L\varphi, \psi)_0 = (L^{1/2}\varphi, L^{1/2}\psi)_0 = (A^2\varphi, \psi)_0 = (\varphi, \psi)_+ = ((\breve{A})^2\varphi, \psi)_0$ for $\varphi, \psi \in \tilde{\mathcal{M}}_+$. Thus $\mathcal{M}_+ = \mathcal{H}_1(L)$ and therefore $\mathcal{M}_+ = \mathcal{H}_2(\breve{A}) = \mathcal{D}(\breve{A})$. □

Thus, we proved theorem 3.1 completely.

5. Discussion and example

From (4.11) and proposition 4.5 it follows that the operators $\breve{A}^2$ and A^2 coincide on the subspace $\tilde{\mathcal{M}}_+ = \mathcal{D}(\breve{A}^2) \cap \mathcal{D}(A^2)$ which is dense in $\mathcal{H}_0$. We note that $\mathcal{D}(\breve{A}^2) \subset \mathcal{D}(A^2)$ and the inclusion $\mathcal{D}(\breve{A}) \subset \mathcal{D}(A)$ is also fulfilled. However, by proposition 5.1, the operators $\breve{A}$, A are distinguished on the infinite-dimensional subspace from $\mathcal{D}(\breve{A}) \cap \mathcal{D}(A)$. Here we encounter one of the deep phenomena of the singular perturbation theory connected with the property of a positive quadratic form to have a lot of closed extensions associated with various self-adjoint operators. Indeed, in our case the operator $\breve{A}^2 \equiv L$ is associated with the closed quadratic form $\nu(\varphi, \psi) := (A\varphi, A\psi)_0, \varphi, \psi \in \mathcal{M}_+$. That is $\nu(\varphi, \psi) = (L\varphi, \psi)_0 = (\breve{A}\varphi, \breve{A}\psi)_0$, where $\breve{A} = \sqrt{L}$. Besides the form ν possesses another closed extensions. One of them is the form $(A\varphi, A\psi)_0, \varphi, \psi \in \mathcal{H}_+$ which is associated with A^2. We note that in general it is not easy to find an explicit action of the operator L and especially $\sqrt{L}$. We partly solved this problem in theorem 4.7 which contains the description of the operator L in the terms of A^2.

We emphasize that L is the Friedrichs extension of $A^2|\tilde{\mathcal{M}}_+$. Therefore, its resolvent admits the evident representation by Krein's formula. This means that in applications our constructions belong to the class of solvable models.

Proposition 5.1. *For the above-defined operator $\breve{A}$, there exists an infinite sequence of orthonormal in $\mathcal{M}_+$ vectors $\{\eta_{+,i}\}_{i+1}^{\infty}$ such that*

$$\breve{A}\eta_{+,i} \neq A\eta_{+,i}, \qquad i = 1, 2, \ldots$$

Proof. We recall that $\mathcal{D}(A) = \mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+, \mathcal{N}_+ = \operatorname{Ker} T$, $\mathcal{D}(\breve{A}) = \mathcal{M}_+ \sqsubset \mathcal{H}_0$, and that the mappings $A : \mathcal{M}_+ \to \mathcal{M}_0 := A\mathcal{M}_+$, $\breve{A} : \mathcal{M}_+ \to \mathcal{H}_0$ are unitary.

Without loss of generality we consider the simplest case, that is, we assume that dim $\mathcal{N}_+ = 1$. Take a unit vector $\eta_+ \in \mathcal{N}_+$ and define $\eta_{+,1} := \breve{A}^{-1}\eta_0$, where $\eta_0 := A\eta_+ \in \mathcal{N}_0 := A\mathcal{N}_+$. Clearly, $\eta_{+,1} \in \mathcal{M}_+$ and $\eta_{+,1} \perp \eta_+$. Therefore

$$\eta_{0,1} := A\eta_{+,1} \perp \eta_0 \qquad \text{and} \qquad A\eta_+ = \breve{A}\eta_{+,1}.$$

Thus, $A\eta_{+,1} \neq \breve{A}\eta_{+,1}$.

Further, we introduce the subspace $\mathcal{M}_{+,1} := \mathcal{M}_+ \ominus \{\eta_{+,1}\}$, where $\{\eta_{+,1}\}$ denotes the one-dimensional subspace spanned by the unit vector $\eta_{+,1}$. Since the map $A : \mathcal{M}_+ \to \mathcal{M}_0$ is unitary we may put $\mathcal{M}_{0,1} := A\mathcal{M}_+ \ominus \eta_{0,1}$, where $\eta_{0,1} = A\eta_{+,1} \in \mathcal{M}_+$. Now we define $\eta_{+,2} := \breve{A}^{-1}\eta_{0,1} \in \mathcal{M}_{+,1}$. Clearly, $\eta_{0,1} \perp \eta_0$ implies $\eta_{+,2} \perp \eta_{+,1}$, where we used that the mapping $\breve{A}^{-1} : \mathcal{M}_+ \to \mathcal{M}_{+,1}$ is unitary. Therefore we have,

$$\mathcal{M}_{0,1} \ni A\eta_{+,2} =: \eta_{0,2} \perp \breve{A}\eta_{+,2} = \eta_{0,1}, \qquad \text{i.e.,} \quad A\eta_{+,2} \neq \breve{A}\eta_{+,2}.$$

In the next step we define $\mathcal{M}_{+,2} := \mathcal{M}_{1,+} \ominus \{\eta_{+,2}\}$. Since the mapping $A : \mathcal{M}_{+,1} \to \mathcal{M}_{0,1}$ is unitary, we have $\mathcal{M}_{0,2} := A\mathcal{M}_{1,+} \ominus \{\eta_{0,2}\}$, where $\eta_{0,2} := A\eta_{+,2} \in \mathcal{M}_{+,1}$. We define $\eta_{+,3} := \breve{A}^{-1}\eta_{0,2} \in \mathcal{M}_{+,2}, \eta_{+,3} \perp \eta_{0,2}$, where we used again that $\breve{A}^{-1}$ is unitary. Therefore

$$\mathcal{M}_{0,2} \ni A\eta_{+,3} \perp \breve{A}\eta_{0,3} = \eta_{0,2}, \qquad \text{i.e.,} \quad A\eta_{+,3} \neq \breve{A}\eta_{+,3}.$$

And so on to any n:

$$A\eta_{+,n} \neq \breve{A}\eta_{+,n}, \qquad \eta_{+,n} := \breve{A}^{-1}\eta_{0,n-1}, \qquad \eta_{0,n-1} := A\eta_{+,n-1}, \quad n > 1.$$ □

We illustrate the above constructions by simple examples.

Let a perturbant T be a rank-1 singular $\mathcal{H}_{-2}$ -class operator acting as follows:

$$T\varphi = \langle \varphi, \omega \rangle_{+,-}\omega, \qquad \varphi \in \mathcal{H}_+ = \mathcal{D}(A),$$

where an element $\omega \in \mathcal{H}_- \backslash \mathcal{H}_{-1}$ is fixed, here $\mathcal{H}_- \equiv \mathcal{H}_{-2}(A)$ (see section 2). So, the set $\mathrm{Ker}\, T \equiv \mathcal{M}_+ = \{\varphi \in \mathcal{H}_+ | \langle\varphi, \omega\rangle_{+,-} = 0\}$ is dense in $\mathcal{H}_0$. Clearly $\mathcal{M}_+ = \mathcal{H}_+ \ominus \{c\eta_+\}$, where $\eta_+ := I_{+,-}\omega = A^{-1}(A^{\mathrm{cl}})^{-1}\omega$. Obviously also that

$$\tilde{\mathcal{M}}_+ = \{\varphi \in \mathcal{D}(A^2) \equiv \mathcal{H}_{++} | \langle\varphi, \omega\rangle_{++,--} = 0\} = \mathcal{M}_+ \cap \mathcal{H}_{++}$$

is dense in $\mathcal{M}_+$ and $\mathcal{H}_0$ too. Now we easily find that the operator L corresponding to the formal sum $A^2 \tilde{+} (\cdot, \omega)\omega$ realizes the Friedrichs extension of the symmetric restriction $\dot{L} := L|\tilde{\mathcal{M}}_+$. This operator has the following explicit description (see proposition 4.6 and cf proposition 3.2 in [4] and the closed result in [21]):

$$\mathcal{D}(L) = P_{\mathcal{M}_+}\mathcal{D}(A^2), \qquad LP_{\mathcal{M}_+}\varphi = A^2\varphi, \qquad \varphi \in \mathcal{D}(A^2).$$

Because the set $\tilde{\mathcal{M}}_+ \subset \mathcal{D}(L)$ is dense in $\mathcal{M}_+$ and $(L\varphi, \psi)_0 = (L^{1/2}\varphi, L^{1/2}\psi)_0 = (A\varphi, A\psi)_0 = (\varphi, \psi)_+, \varphi, \psi \in \tilde{\mathcal{M}}_+$ we conclude that $\mathrm{dom}\, L^{1/2} \equiv \mathrm{dom}\, \breve{A} = \mathcal{M}_+$. However, it is a non-trivial problem to describe in a general case the action of the operator $\breve{A} = L^{1/2}$. We are able to solve this problem only in the concrete examples.

Let $\mathcal{H}_0 = L_2(\mathbf{R}, \mathrm{d}x)$, $A = -\frac{\mathrm{d}^2}{\mathrm{d}^2 x}$, and $\omega = \delta_0(x)$ be the Dirac delta function supported in the origin of coordinates. So, $\mathcal{H}_+ = W_2^2(\mathbf{R}, \mathrm{d}x)$, $\mathcal{H}_{++} = W_2^4(\mathbf{R}, \mathrm{d}x)$, where $W_2^d(\mathbf{R}, \mathrm{d}x)$ stands for the usual Sobolev space of order d. Now $\mathcal{M}_+ = \{\varphi \in W_2^2(\mathbf{R}, \mathrm{d}x) | \varphi(0) = 0\}$ and $\tilde{\mathcal{M}}_+ = \{\varphi \in W_2^4(\mathbf{R}, \mathrm{d}x) | \varphi(0) = 0\}$.

Then the operator $L = \frac{\mathrm{d}^4}{\mathrm{d}^4 x} \tilde{+} \delta_0$ which is the Friedrichs extension of $\dot{L} := L|\{\varphi \in W_2^4(\mathbf{R}, \mathrm{d}x) | \varphi(0) = 0\}$ admits the following description:

$$L\varphi = \frac{\mathrm{d}^4}{\mathrm{d}^4 x}\varphi, \qquad \mathcal{D}(L) = \{\varphi \in W_2^4(\mathbf{R}\backslash\{0\}, \mathrm{d}x) \cap W_2^3(\mathbf{R}, \mathrm{d}x) | \varphi(0) = 0)\}.$$

Moreover, coming to the spectral representation we find the explicit formula for the action of $\breve{A} \equiv L^{1/2}$:

$$\breve{A}\varphi(x) = -\varphi''(x) + 2/\pi \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + y^2)(|x| + |y|)} \varphi''(y)\, \mathrm{d}y,$$

where the domain

$$\mathcal{D}(\breve{A}) = \{\varphi \in W_2^2(\mathbf{R}, \mathrm{d}x) | \varphi(0) = 0\}.$$

We observe that $\breve{A}\varphi = -\varphi''$ if φ is an odd function. And $\breve{A}\varphi \neq -\varphi''$ on the infinite-dimensional subspace of all even functions (cf with proposition 5.1).

Finally we note that operators of type $(-\Delta)^n$, $n > 1$ have an evident physical sense in resonator theory. Moreover, the model $(-\Delta)^2 + \delta_0$ admits the explicit solution and therefore belongs to the class of solvable models. In particular, the corresponding perturbed operator has the representation via Krein's resolvent formula. However, the highly non-trivial problem is to find an explicit view for the square root $\sqrt{(-\Delta)^2 + \delta_0}$. We solve this problem in the above example.

Acknowledgments

This work was partly supported by the projects: INTAS 00-257, DFG 436 UKR 113/67 and DFG 436 UKR 113/78. The author thanks Professor L Nizhnik for the discussion of the concrete examples.

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